ON THE STABILIZATION OF STOCHASTIC SYSTEMS

(O STABILIZATSII STOKHASTICHESKIKH SISTEM)

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This paper is concerned with the study of control systems in which the transition process is described by means of stochastic linear differential equations. The construction of the Liapunov functions is accomplished by means which generate the known methods of Chetaev [1]. The system is subjected to the action of a random effect of the Markov type [2], developed during the control process, and also to disturbances which have the character of random external impulsive disturbances [3]. The problem of the stabilization of such systems generalizes Liapunov's [4] stability problem. The problem considers the establishment of the control action of a control element (control program) which assures statistical stability of a given motion with arbitrary initial deviations. Such a control program will be called "allowable". Sufficient conditions for the existence of an allowable control program in an nth order linear system are established. For a system of second order, the possibility of constructing allowable control is established by means of a geometric interpretation in the phase plane. The existence criteria given in this paper help explain the problem of the establishment of the optimum control action which minimizes some integral criterion on the quality of the transition process.

1. Statement of problem. 1. Let us study a stationary control system where the transition process is described by the vector equation

$$dx / dt = A(\eta)\, x + c(\eta)\, \xi + \gamma(x) \qquad (1.1)$$

$$\xi = \xi(x_1, \ldots, x_n, \eta) \qquad (1.2)$$

Here $x = \{x_1, \ldots, x_n\}$ denotes an n-dimensional vector of the coordinates x_i which are equal to zero in a given unperturbed motion; ξ is a scalar governing the control action; $A(\eta)$ is a matrix of the form $\|a_{ij}\|_1^n$, whose coefficients are functions of the random factor $\eta(t)$; $c = \{c_1, \ldots c_n\}$ denotes an n-dimensional vector which depends on the random quantity

$\eta(t)$, and $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ is an n-dimensional disturbance vector.

We assume that the sought relation $\xi(x, \eta)$ is linear:

$$\xi(x, \eta) = \beta_1(\eta) x_1 + \ldots + \beta_n(\eta) x_n \quad (1.3)$$

The problem calls for the establishment of a stabilizing control program ξ by means of choosing the functions $\beta_1(\eta), \ldots, \beta_n(\eta)$. The coefficients of the matrix $a_{ij}(\eta)$ and the components of the vector $c_i(\eta)$ are known. A description of the statistical properties of the random quantity $\eta(t)$ and the external perturbation $\gamma(x)$ is given below.

2. We shall describe the random quantity $\eta(t)$ in the following manner [2, p.234]. Assume that $\eta(t)$ is a unique Markov process. The change of $\eta(t)$ takes place in a closed finite interval $\eta_1 \leqslant \eta(t) \leqslant \eta_2$.

We denote by $P[Q/L]$ the probability of an event Q occurring under the condition L. For a sufficiently small interval of time Δt we have

$$P[\eta(t + \Delta t) = \alpha / \eta(t) = \alpha] = 1 - q(\alpha) \Delta t + o(\Delta t)$$
$$P[\eta(t + \Delta t) \neq \alpha, \eta(t + \Delta t) \leqslant \beta / \eta(t) = \alpha] = q(\alpha, \beta) \Delta t + o(\Delta t)$$

where $q(\alpha)$ and $q(\alpha, \beta)$ are assumed to be known.

If it is assumed that the function $q(\alpha, \beta)$ has a density $p(\alpha, \beta)$, then

$$P[\beta_1 \leqslant \eta(t + \Delta t) \leqslant \beta_2, \ \eta(t + \Delta t) \neq \alpha / \eta(t) = \alpha]$$
$$= \Delta t \int_{\beta_1}^{\beta_2} p(\alpha, \beta) \, d\beta + o(\Delta t)$$

Under these conditions of realization $\eta^p(t)$ of the process the $\eta(t)$ are step functions [2, p.233].

3. The perturbation $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a random function which describes a constantly acting disturbance. We assume that γ can be represented in the form

$$\gamma = \mu \psi(t), \quad \psi(t) = \sum_k \nu_k \delta(t - t_k) \quad (1.4)$$

Here $\psi(t)$ is an impulsive function which depends on the random variables ν_k and t_k, where the values of ν_k are independent for various k and are also independent of t_k; the symbol $\delta(t)$ denotes the Dirac delta function; $\mu = \{\mu_1, \ldots, \mu_n\}$ is a constant vector or a vector function $\mu(x)$.

For the random variable t_k the Poisson distribution with a frequency λ [3, p.63] is satisfied.

With such a description of the disturbance, the realization of the solution $x^p(t)$ is subject to a discontinuous change $\{\Delta_u x_i\}$ at the instant t_k. If the function $x_i^p(t)$ is assumed to be continuous from the right, one can compute

$$\Delta_u x_i = x_i(t_k) - x_i(t_k - 0) = \mu_i \nu_i \qquad (1.5)$$

Suppose the mean value of ν_i is equal to zero ($M\{\nu_i\} = 0$, $k = 1, \ldots, m$); the scattering $M\{\nu_i^2\} = \sigma_i^2 \geqslant 0$ is known as well as the correlation coefficients k_{ij}, which describe the statistical coupling of γ_i and γ_j ($M\{\nu_i \nu_j\} = k_{ij}\sigma_i\sigma_j$). We shall analyze the following cases:

α) The disturbing action diminishes to zero as the given law of motion approaches $x = 0$, where

$$\mu_i = \mu_{i1}x_1 + \ldots + \mu_{in}x_n \qquad (i = 1, \ldots, n, \quad \mu_{ij} = \text{const}) \qquad (1.6)$$

β) The disturbance does not depend on $x(t)$

$$\mu_i = \text{const} \qquad (1.7)$$

4. The solution of the system (1.1) will be a random vector function $\{x(t), \eta(t), \gamma(t)\}$ whose realizations $\{x^p(t), \eta^p(t), \gamma^p(t)\}$ satisfy Equation (1.1) for $\eta = \eta^p(t)$, $\gamma = \gamma^p(t)$.

As a result of the disturbance action the realization of the solution $x^p(t)$ is discontinuous, which introduces some peculiarities into further considerations.

Thus, each initial condition x_0, η_0 generates a random Markov function $\{x(t), \eta(t)/x_0, \eta_0, t_0 = 0\}$.

5. We shall introduce some definitions which modify the concepts presented in [5,6].

Definition 1.1. The solution $x = 0$ of system (1.1) will be said to be stable in the probability sense if for any arbitrarily small numbers $\epsilon > 0$ and $p > 0$ one can determine a $\delta > 0$ such that the following inequality holds:

$$P\left[\sum_n x_i^2(t) < \varepsilon^2 \quad \text{for } t > t_0 \Big/ \sum_n x_{i0}^2 \leqslant \delta^2 \quad \text{for } t = t_0\right] > 1 - p$$

Here, on the left-hand side stands the probability of the inequality

$$\sum_n x_i^2(t) < \varepsilon^2 \text{ for the initial conditions } \sum_n x_{i0}^2 \leqslant \delta^2$$

If, in addition, for any $\omega > 0$ the following condition is satisfied for each initial value x_0

$$\lim P\left[\sum_n x_i^2(t) < \omega^2\right] = 1 \quad \text{as } t \to \infty$$

then the solution will be called asymptotically stable in the probability sense (as a whole).

Definition 1.2. The solution $x(t)$ of the system (1.1) will be said to be stable in the mean if for an arbitrary number $\epsilon > 0$ one can determine a $\delta > 0$ such that the following inequality holds:

$$M\{x_1^2(t) + \ldots + x_n^2(t);\ x(t),\ \eta(t) / x_0, \eta_0\} < \varepsilon^2 \text{ for } x_{10}^2 + \ldots + x_{n0}^2 \leqslant \delta^2$$

Here, on the left-hand side $M\{x_1^2(t) + \ldots + x_n^2(t)\}$ is the mathematical expectation with the initial conditions x_0, η_0, $x^2_{10} + \ldots + x^2_{n0} \leqslant \delta^2$.

If, in addition, for each initial value of x_0 the condition

$$\lim M\{x_1^2(t) + \ldots + x_n^2(t)\} = 0 \quad \text{as } t \to \infty$$

is satisfied, the solution can be said to be asymptotically stable in the mean (as a whole).

Using Definitions 1.1 and 1.2, we can formulate requirements leading to the allowable control program. We shall call the control program $\xi(x, \eta)$ allowable if the following conditions are fulfilled:

a) Solution $x = 0$ is on the average asymptotically stable with respect to arbitrary initial deviations.

b) The integral of the mean square error [6, p.428]

$$\int_0^\infty M\left\{\sum_n x_i^2 + \xi^2\right\} dt$$

is finite for arbitrary initial conditions x_0, η_0.

c) The system possesses stability of the given motion $x = 0$ in the probability sense.

Note 1.1. In the case (β) of Section 3 where, on account of the presence of a disturbance, the unperturbed motion $x = 0$ is impossible to be realized without some unavoidable error, whose value is determined in the space $\{x_1, \ldots, x_n\}$ by means of the inequality $x_1^2 + \ldots + x_n^2 < \Delta^2$, we shall introduce the concept of (p, Δ) – allowable control. This is a control program which satisfies the conditions

$$(\alpha) \qquad \lim [M\{x_1^2(t) + \ldots + x_n^2(t)\}] < \Delta^2 \quad \text{as} \quad t \to \infty,\ x = x_{i0};\ i = 1, \ldots, n$$

$$(\beta) \qquad \int_0^\infty M_\Delta \left\{ \sum_n x_i^2 + \xi^2 / x_0, \eta_0 \right\} dt < \infty$$

Here, the symbol M_Δ denotes the mathematical expectation in terms of the random variables $x_i(t)$, $\xi(t)$ which satisfy the inequality

$$\Sigma x_i^2(t) + \xi^2(t) > \Delta^2$$

(γ) For a given number $p > 0$ one can find a $\rho > 0$ such that

$$P\,[x_1^2(t) + \ldots + x_n^2(t) < \Delta^2 \quad \text{for} \quad t > t_0 / x_{10}^2 + \ldots + x_{n0}^2 \leqslant \rho^2] > 1 - p$$

It can be seen from the conditions (α) to (γ) that the stability in the probability sense and the stability in the mean have somewhat different meanings than in the Definitions 1.1 and 1.2.

2. Approach to solution. 1. Suppose that in Equation (1.1), the perturbation is $\gamma(x) = 0$. Then the realization $x^p(t)$ of the solution of the system (1.1) will be a piecewise-smooth continuous function, and on the basis of Theorem 6.1 [5, p.818] the existence criterion of the allowable control is determined.

If a positive-definite quadratic form $v(x, \eta)$ can be found whose derivative $dM\{v\}/dt$, formed on the basis of Equation (1.1) with $\xi = \beta_1(\eta)x_1 + \ldots + \beta_n(\eta)x_n$, $|\beta(\eta)| < b = \text{const}$ satisfies the condition

$$dM\{v\} / dt = -w(x, \eta) \tag{2.1}$$

where $w(x, \eta)$ is a negative-definite quadratic form, then the function $\xi(x, \eta)$ gives the allowable control.

Condition (2.1), according to the above-mentioned theorem, assures the asymptotic stability in the mean and the stability in the probability sense of the solution $x = 0$. The convergence of the integral of the mean square error follows from the equation

$$v(x_0, \eta_0) = \int_0^\infty M\{w(x, \eta) / x_0, \eta_0\}\, dt$$

which is obtained by means of integrating the averaged equation (2.1) with respect to t, and from the ability to choose the constant L for which $x_1^2 + \ldots + x_n^2 + \xi^2 \leqslant Lw(x, \eta)$ holds (for detail see an analogous discussion in [6]).

Note 2.1. References [5,6] used a description of the random variable $\eta(t)$ in terms of a transition matrix $\|p_{ij}\|_1^n$. The more general method of describing $\eta(t)$ in the present paper does not introduce any significant changes into the reasoning.

Analogously to [6, p.426], the derivative $dM\{v\}/dt$ is computed in the following manner:

$$\frac{dM\{v\}}{dt} = \sum_{i=1}^{n} \frac{\partial v(x, \eta)}{\partial x_i}\left[c_i(\eta)\xi + \sum_{j=1}^{n} a_{ij}(\eta)x_j\right] + \int_{\tau_1}^{\tau_2} [v(x, \beta) - v(x, \eta)]\, d_\beta q(\eta, \beta) \quad (2.2)$$

Here the Stieltjes integral replaces the summation in Equation (4.6) [6, p.427].

2. Let us assume that in the absence of disturbances, there exists an allowable control. Considering the perturbation $\gamma(x)$ the following additional terms appear in the expression $dM\{v\}/dt$:

$$\frac{dM\{v\}}{dt} = \left[\frac{dM\{v\}}{dt}\right]_1 + \frac{\lambda}{2}\sum_{i,j}^{n} \frac{\partial^2 v}{\partial x_i \partial x_j} k_{ij}\sigma_i\sigma_j\mu_i\mu_j \quad (2.3)$$

The negative-definite form $[dM\{v\}/dt]_1$ is computed by means of Formula (2.2). Let us explain how the second term of the right-hand side of (2.3) is obtained.

Assume that during a sufficiently small time interval $\Delta t > 0$ the coordinate x_i is subjected to only one jump, whose magnitude, computed from Formula (1.5), is $\Delta_u x_i = \mu_i \nu_i$. In the time interval $t \leqslant \tau < t + \Delta t$ we shall neglect the continuous variations of the function $v(x, \eta)$ with x and shall assume that $\eta(t) = a = \text{const}$. These assumptions give an error which is small to a higher order and vanishes as $\Delta t \to 0$. Then the mathematical expectation of the change of v with $\{\Delta_u x_i\}$ has the form

$$M\{\Delta v\} \approx M_\nu\{\Delta_u v\} P[u]$$

The symbol M_ν denotes the mathematical expectation of $\Delta_u v$, which was obtained by averaging over the random variables ν_i; $P[u]$ denotes the probability of one impulse occurring in the interval Δt. We have

$$P[u] = \lambda\Delta t + o(\Delta t)$$

For the computation of $M_\nu\{\Delta_u v\}$ we use the Taylor expansion of the quadratic form $\Delta_u v$; here all terms higher than second order are equal to zero. Thus

$$M\{\Delta v\} \approx M_\nu\left\{\sum_{i=1}^{n} \frac{\partial v(x, \alpha)}{\partial x_i} \Delta_u x_i + \frac{1}{2} \sum_{i,j}^{n} \frac{\partial^2 v(x, \alpha)}{\partial x_i \partial x_j} \Delta_u x_i \Delta_u x_j\right\} \lambda \Delta t \qquad (2.4)$$

In this equation

$$M_\nu \{(\partial v / \partial x_1) \Delta_u x_1 + \ldots + (\partial v / \partial x_n) \Delta x_n\} = 0$$

since the mean value of ν_i is equal to zero. Thus, taking into account the equation $M_\nu\{\nu_i \nu_j\} = k_j \sigma_i \sigma_j$, we obtain from (2.4) after division by Δt and passing to the limit as $\Delta t \to 0$ the desired expression for the terms of the right-hand side of (2.3), which are determined by the disturbance.

Note that the second term of the right-hand side of (2.3) is positive, since $v(x, \eta)$ is a positive-definite quadratic form.

It is impossible to use directly the results of [5] in the case of the disturbance, since the realizations $x^p(t)$ are discontinuous.

Furthermore, in the case 1 (β) there exists a neighborhood of the origin $x_1^2 + \ldots + x_n^2 \Delta^2$, where the derivative $dM\{v\}/dt$ is known to be positive.

In order to extend the conclusions made in Section 2 (1) to the case of the action of a disturbance, we shall formulate the following theorem:

Theorem 2.1. Suppose that in Equations (1.1) the disturbance γ does not depend on x (the vector μ is constant) and that there exists a positive-definite quadratic form $v(x, \eta)$, whose derivative of the mathematical expectance computed by means of Formula (2.2) for $\xi = \xi(x, \eta)$ is negative-definite. Then for arbitrary $p > 0$ and $\Delta > 0$ one can find numbers κ and H such that when satisfying the inequalities

$$\lambda \sigma_i^2 < \varkappa \qquad |\Delta_u x_i| < H \qquad (2.5)$$

the control program $\xi(x, \eta)$ is (p, Δ) -allowable.

The proof of Theorem 2.1) proceeds in a manner analogous to the statement in [5] (p. 823). The difference in reasoning and the presence of the limitation $|\Delta_u x_i| < H$ is caused by the discontinuity of the realization $x^p(t)$. The details introduced into the reasoning by this difference are omitted here.

If the vector μ depends linearly on x (case 1 (α)), then because of the action of the disturbance, there appears on the right-hand side of Formula (2.3) a positive quadratic form, whose coefficients are determined by the values of the dispersion σ_i and the coefficients μ_{ij} in

(1.6). The existence criterion of the allowable control, similar to Theorem 2.1, is formulated as follows.

Assume that in Equation (1.1) the disturbance y varies linearly (1.6) with x, and that there exists a positive-definite form $v(x, \eta)$ whose derivative of mathematical expectance, computed by means of Formula (2.2) with $\xi = \xi(x, \eta)$, is negative-definite. Then one can always find $\rho > 0$ and $s > 0$ such that when the inequalities

$$\lambda\sigma_i^2 < \rho, \qquad |\mu_{ij}| < s \tag{2.6}$$

are satisfied, the control $\xi(x, \eta)$ is allowable.

3. Stability in the overall probability sense. Suppose a transition process is described by means of a stochastic equation in vector form

$$dx / dt = A(\eta) x + c(\eta) \xi \tag{3.1}$$

It will be said that stability in the overall probability sense will exist if the solutions $x(t) = 0$ of Equation (3.1) correspond to the requirements of Definition 1.1, and in addition such an $r_1 > 0$ can be found for two arbitrary numbers $p > 0$ and $r_0 > 0$ that the following inequality holds:

$$P\left[\sum_n x_i^2(t) < r_1^2 \ \text{ for } t > t_0 \Big/ \sum_n x_{i0}^2 \leqslant r_0^2 \ \text{ for } t = t_0\right] > 1 - p$$

This section presents conditions with the aid of which one can construct a control program $\xi(x, \eta)$ which assures such a stability.

If the system of vectors $c(\eta)$, $A(\eta)c(\eta)$, ..., $A^{n-1}(\eta)c(\eta)$ is linearly independent, then for an arbitrary fixed value of $\eta = \eta_1$ in the form of a Liapunov function an optimum control time $T^\circ(x_1, \ldots, x_n)$ can be adopted, which can be determined from the conditions [7]

$$\min \int_0^{T^\circ} \left|\sum_n l_i h_i(\tau)\right| d\tau = 1, \qquad \sum_n l_i x_i = -1, \quad h_i(\tau) = \sum_{k=1}^{n} f_{ik}(\tau) c_k$$

Here, $f_{ik}(\tau)$ are the coefficients of the inverse matrix of the fundamental system of solutions of the equation

$$dx / dt = Ax \tag{3.2}$$

The optimum control program $u(x)$ from the point of view of fast action can be found by the method described in [7].

Under the given conditions function $T^\circ(x) = v(x)$ exists, and along the optimum trajectory $dv(x)/dt = -1$ [7, p.634]. We shall find a control program $\xi = u(x, \eta)$ which optimizes the system with a chosen value of η and the functions $v(x, \eta)$ for each fixed η.

Assuming $v(x, \eta)$, constructed in such a manner, to be a function of the random variable $\eta(t)$, we compute the derivative of mathematical expectance $dM\{v\}/dt$ on the basis of Equation (3.1).

Taking into account (2.2) we obtain

$$\frac{dM\{v(x, \eta)\}}{dt} = -1 + \int_{\eta_1}^{\eta_2} [v(x, \beta) - v(x, \eta)]\, d_\beta q(\eta, \beta) \tag{3.3}$$

The integral in (3.3) can be positive as well as negative. Thus, one should find such a neighborhood $x_1^2 + \ldots + x_n^2 < \epsilon^2$ around the origin where the right-hand side of (3.3) will be certainly negative because of the smallness of T°. Of course, $\epsilon > 0$ is determined by the maximum of the integral on the right-hand side of (3.3).

In this neighborhood we choose a control program $\xi = u(x, \eta)$, assuming it to be a function of the random quantity $\eta(t)$. It will be shown that the constructed control program can be extended over the entire space in order to assure the stability of the solution $x(t)$ in the overall probability sense.

Let $p > 0$ be a given small number. Since in the region $x_1^2 + \ldots + x_n^2$ $v(x, \eta)$ satisfies the conditions of the theorem on the stability of the solution $x = 0$ in the probability sense [5, p.812], one can find a $\delta > 0$ such that for the condition $x_{10}^2 + \ldots + x_{n0}^2 \leqslant \delta^2$ the following inequality is fulfilled:

$$P\,[x_1^2(t) + \ldots + x_n^2(t) < \varepsilon^2] > 1 - p \tag{3.4}$$

Let us transform by means of a uniform expansion of the space $\{x_1, \ldots, x_n\}$ the sphere $x_1^2 + \ldots + x_n^2 = \rho^2$ into the sphere $x_1^2 + \ldots + x_n^2 = \delta^2$. Then the region $\delta^2 \leqslant x_1^2 + \ldots + x_n^2 < \epsilon^2$ transforms into the region

$$\varepsilon^2 \leqslant x_1^2 + \ldots + x_n^2 < \varepsilon_1^2 \qquad (\varepsilon_1 = \varepsilon^2/\delta)$$

It is required that the function $u(x, \eta)$ constructed above transform at the point $\{x\}$ of the region $\delta^2 \leqslant x_1^2 + \ldots + x_n^2 < \epsilon^2$ into the function u_1 in a linear fashion

$$u_1\left(\frac{\varepsilon}{\delta}x\right) = \frac{\varepsilon}{\delta}u(x)$$

In the region $\epsilon^2 \leqslant x_1^2 + \ldots + x_n^2 < \epsilon_1^2$ we construct a function V_1 such that $V_1(x) = (\epsilon/\delta)^2 \times v(x\delta/\epsilon)$. Then grad $V_1 = (\epsilon/\delta)$ grad v, and consequently, taking into consideration the linearity of the system (3.1), we have (for $\xi = u_1$ in the region $\epsilon^2 \leqslant x_1^2 + \ldots + x_n^2 < \epsilon_1^2$)

$$\frac{dV_1(x\varepsilon/\delta, \eta)}{dt} = \frac{\varepsilon^2}{\delta^2}\frac{dv(x, \eta)}{dt}, \quad \text{or} \quad \frac{dV_1}{dt} = -\left(\frac{\varepsilon}{\delta}\right)^2 \tag{3.5}$$

(since $dv/dt = -1$). By continuing a similar construction process, one can find a series of consecutive concentric spheres with radii ϵ_j of functions V_j and u_j determined in the layer $\epsilon_{j-1}^2 \leqslant x_1^2 + \ldots + x_n^2 < \epsilon_j^2$. It follows from (3.4) and (3.5) that for an arbitrary sphere ϵ_j the fulfilment of the inequality $P[x_1^2(t) + \ldots + x_n^2(t) < \epsilon_j^2] > 1 - p$ is assured if only the initial deviations are bounded by the region $x_1^2 + \ldots + x_n^2 < \epsilon_{j-1}^2$.

It will be shown that the constructed control program assures stability in the overall probability sense for the system (3.1).

Let $\zeta > 0$ and $q > 0$ be given numbers. Assume that the region $x_1^2 + \ldots + x_n^2 < \zeta^2$ lies inside the kth sphere with a radius $r = \epsilon_k$. Of course, the probability of the realization $x^p(t)$ going beyond the sphere of radius ϵ_{k+1} with the initial conditions $x_{10}^2 + \ldots + x_{n0}^2 < \zeta^2$ will not be greater than p. One can choose an integer $l > 0$ such that $p^l \leqslant q$. Then for the sphere of radius ϵ_{k+1} the following inequality holds:

$$P\,[x_1^2(t) + \ldots + x_n^2(t) < \varepsilon_{k+l} \,/\, x_{10}^2 + \ldots + x_{n0}^2 \leqslant \zeta^2] > 1 - q$$

which also proves the stated hypothesis on the stability in the overall probability sense.

Thus, if the system of vectors $c(\eta)$, $A(\eta)c(\eta)$, $\ldots$, $A^{n-1}(\eta)c(\eta)$ is linearly independent, then it is possible to construct a control program which assures the stability of the system (3.1) in the overall probability sense

4. System of *n*th order. Sufficient conditions for the formation of allowable control. 1. Let us continue the study of Equation (3.1). Suppose that the function $q(a, \beta)$, which characterizes the probability of the transition $(\eta = a) \to (\eta = \beta)$, has the density $p(a, \beta)$. The variation of $\eta(t)$ is limited by the closed interval

$\eta_1 \leqslant \eta(t) \leqslant \eta_2$.

Theorem 4.1. If for any value of $\eta(t)$ on the segment $\eta_1 \leqslant \eta(t) \leqslant \eta$ the following conditions are fulfilled:

1) the vector system $c(\eta)$, $A(\eta)c(\eta)$, ..., $A^{n-1}(\eta)c(\eta)$ is linearly independent;

2) there exist finite estimates of the coefficients of the matrix $A(\eta)$, the components of the vector $c(\eta)$ and the density $p(\alpha, \beta)$

$$\left|\frac{\partial a_{ij}}{\partial \eta}\right| < N, \qquad \left|\frac{\partial c_i}{\partial \eta}\right| < N, \qquad p(\alpha, \beta) < \frac{L}{|\alpha-\beta|} \tag{4.1}$$

then one can find a constant $Q > 0$ such that when the inequality $NL < Q$ is satisfied, it is possible to construct an allowable control which stabilizes the system (3.1).

Proof. From the first condition of the theorem it follows that for each fixed value of $\eta = \eta_\phi$ one can construct a control program $\xi^\circ(x, \eta_\phi)$ which assures the asymptotic stability and minimizes the mean-square integration error [8]:

$$V(x, \eta_\varphi) = \int_0^\infty \left[\sum_n x_i^2 + \xi^2\right] dt$$

We have [9]

$$\left(\frac{dV}{dt}\right)_{\xi^\circ} + \sum_n x_i^2 + \xi^{\circ 2} = \min_\xi \left[\frac{dV}{dt} + \sum_n x_i^2 + \xi^2\right] = 0 \tag{4.2}$$

By known methods and by inserting the control program $\xi^\circ(x, \eta_\phi)$ into (4.2) we obtain an equation for $V(x, \eta_\phi)$:

$$-\frac{1}{4}\left[\sum_{i=1}^n c_i \frac{\partial V}{\partial x_i}\right]^2 + \left[\sum_{i=1}^n \left(\frac{\partial V}{\partial x_i} \sum_{j=1}^n a_{ij} x_j\right)\right] = -\sum_{i=1}^n x_i^2 \tag{4.3}$$

In order to clarify the dependence of the constructed function V upon η we differentiate (4.3) (for the present formally) with respect to η, denote $\partial V(x, \eta)/\partial \eta = \alpha$ and transpose all terms containing α and $\partial \alpha/\partial \eta$ to the left-hand side of the equation. Thus, we obtain the equation

$$-\frac{1}{2}\left[\sum_{i=1}^n c_i \frac{\partial V}{\partial x_i}\right]\left[\sum_{i=1}^n c_i \frac{\partial \alpha}{\partial x_i}\right] + \sum_{i=1}^n \left(\frac{\partial \alpha}{\partial x_i} \sum_{j=1}^n a_{ij} x_j\right)$$
$$= -\frac{1}{2}\left[\sum_{i=1}^n c_i \frac{\partial V}{\partial x_i}\right]\left[\sum_{i=1}^n \frac{\partial c_i}{\partial \eta} \frac{\partial V}{\partial x_i}\right] + \sum_{i=1}^n \left(\frac{\partial V}{\partial x_i} \sum_{j=1}^n \frac{\partial a_{ij}}{\partial \eta} x_j\right) \tag{4.4}$$

Since $\xi^\circ = -1/2(c_1\partial V/\partial x_1 + \ldots + c_n\partial V/\partial x_n)$, the left-hand side of (4.4) is the total derivative $d\alpha(x, \eta)/dt$ on the basis of system (3.1) with $\xi = \xi^2(x, \eta)$, and the right-hand side is a quadratic form. With fixed η the system (3.1) is asymptotically stable. Consequently [10, p.61] one can assert that there exists a unique solution of (4.4) which is a quadratic form of $\alpha(x, \eta)$. The conditions of the theorem are fulfilled on the closed interval $\eta_1 \leqslant \eta \leqslant \eta_2$, and the form coefficients of α depend continuously on η. Therefore, a constant $l > 0$ can be chosen such that the following estimate is fulfilled:

$$|\alpha| = \left|\frac{\partial V(x, \eta)}{\partial \eta}\right| \leqslant Nl \sum_n x_i^2(t) \tag{4.5}$$

Let us choose $V(x, \eta)$ as a Liapunov function, where η is the random magnitude of $\eta(t)$. Because of (3.1) the derivative of the mathematical expectation $dM\{v\}/dt$ consists of a sum of numbers which consider the change of the function v over all arguments, including $\eta(t)$ (Section 2):

$$\frac{dM_t\{v\}}{dt} = -\sum_n x_i^2 - \xi^2 + \int_{\eta_1}^{\eta_2} [v(x, \beta) - v(x, \alpha)]\, p(\alpha, \beta)\, d\beta \tag{4.6}$$

Using the values from (4.1) and (4.5) we obtain

$$\left|\int_{\eta_1}^{\eta_2} [v(x, \beta) - v(x, \alpha)\, p(\alpha, \beta)\, d\beta\right| \leqslant NLl\,|\eta_2 - \eta_1| \sum_n x_i^2(t) \tag{4.7}$$

To assure stability and to satisfy the statements of Theorem 4.1, it is sufficient to require that the right-hand side of (4.6) be negative-definite, i.e. that the inequality

$$NL \leqslant \frac{1}{l\,|\eta_2 - \eta_1|} = Q \tag{4.8}$$

be fulfilled.

Note that the constant l is determined from the solution of known problems with fixed η from the interval $\eta_1 \leqslant \eta \leqslant \eta_2$, and consequently the value of Q can be calculated.

Note 4.1. The condition $p(\alpha, \beta) \leqslant L/|\alpha - \beta|$ corresponds to the requirement of a small probability of large jumps with varying $\eta(t)$, i.e. the inequality $NL < Q$ means that the probability of large variations of the system (3.1) is limited.

For the stochastic system, this result corresponds to the well-known result of Chetaev on the freezing of variable coefficients in ordinary differential equations [1].

Note 4.2. If the conditions of Theorem (4.1) are fulfilled and allowable control exists, then one can also construct an optimum control program which minimizes the integral mean-square evaluation of the quality in the stochastic system (3.1) (see, for instance [6, p.428]). Thus, the following derivation is the result of the preceding.

From the already-established existence conditions of the allowable control, it follows that if the vector system $c(\eta)$, $A(\eta)c(\eta)$, ..., $A^{n-1}(\eta)c(\eta)$ is linearly independent, and the evaluations in (4.1) are carried out, one can construct an optimum control program which optimizes the system (3.1) according to some integral criterion of the quality of the transition process.

2. Let us study the case of the action of a disturbance $\gamma(x)$, when μ_i = const (case (β), Section 1).

Suppose the conditions of Theorem 4.1 are fulfilled, and in the absence of the disturbance the system is stabilized by means of a control program $\xi(x, \eta)$. We construct a function $V(x, \eta)$ just as in the proof of Theorem 4.1.

Then the expression for the derivative $dM\{v\}/dt$, computed considering the disturbing action γ, differs from (4.6) by components of the form

$$S = \frac{\lambda}{2}\sum_{ij}^{n}\frac{\partial^2 v}{dx_i dx_j}K_{ij}\sigma_i\sigma_j\mu_i\mu_j$$

on the right-hand side (subsection 2, Section 2). Since $V(x, \eta)$ is a positive-definite quadratic form, the relation

$$S \leqslant G \tag{4.9}$$

exists, which determines the region of non-correctible error $x_1^2 + \ldots + x_n^2 < \Delta^2$.

In order to stabilize the system by means of the control program $\xi(x, \eta)$, which in the case under study is (p, Δ)-allowable, it is sufficient to require that the limitations (2.5) on Theorem (2.1) be fulfilled. This determines the existence conditions of the control program.

If μ_i depends linearly on $\{x_1, \ldots, x_n\}$ (case a, Section 1), then on the left-hand side of (4.9) a quadratic form like

$$\sum_{i,j}^{n} g(\eta)\, x_i x_j$$

is obtained.

The stabilization of such a system is possible with the aid of the allowable control, if the evaluations of (2.6) are fulfilled, where in choosing the number $s > 0$ in (2.6) (as well as the number $H > 0$ in (2.5)) a discontinuous character of the realization is assumed.

5. System of the second order. Sufficient conditions for the formulation of allowable control. In a system where the transition process is described by second-order equations, one can find sufficient existence conditions for the allowable control $\xi(x, \eta)$ by means of a geometrical interpretation in the phase plane $\{x_1, x_2\}$. These conditions differ somewhat from the requirements of the general theorem 4.1 since the limit on the rate of change of the random variable $\eta(t)$ (see Note 4.1) is not used here.

Let us analyze this vector equation

$$\frac{dx}{dt} = k + c\xi, \qquad k = \left\{\sum_{i=1}^{2} a_{1i}(\eta)\, x_i, \sum_{i=1}^{2} a_{2i}(\eta)\, x_i\right\} \tag{5.1}$$

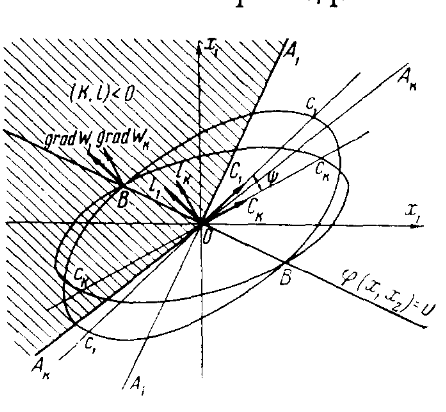
Here k is the vector of the right-hand side of the fundamental linear system

$$c = \{c_1(\eta), c_2(\eta)\}$$

It will be shown that under certain conditions one can establish the allowable control

$$\xi = m(\beta_1 x_1 + \beta_2 x_2) \qquad (m, \beta_1, \beta_2 = \text{const}) \tag{5.2}$$

We fix the variable $\eta = \eta_1$. In the x_1x_2-plane (see figure) we construct the vector $c_1 = c(\eta_1)$. We denote by $l_1 = \{-c_{12}, c_{11}\}$ a vector equal in magnitude and perpendicular to c_1.



Suppose the scalar product (k, l_1) along the line A_1A_1 is equal to zero. The line A_1A_1 separates the plane into regions of negative and positive values of the product (k, l_1).

We assume that for all values of $\eta_1 \leqslant \eta \leqslant \eta_2$ the directions of the vector $c(\eta)$ lie within the angle ψ between c_1 and c_k.

Suppose that there exists a general (for any η) negative region $(k, l) < 0$ (such a case is shown in the illustration) where the vector l for $\eta_1 \leqslant \eta \leqslant \eta_2$ is directed toward that region.

We choose β_1, β_2 in (5.2) such that the line $\phi(x_1, x_2) = \beta_1 x_1 + \beta_2 x_2 = 0$ passes through the region $(k, l) < 0$. Let us study now the positive-definite form with constant coefficients

$$w_1 = a_{11}^{(1)} x_1^2 + 2a_{12}^{(1)} x_1 x_2 + a_{22}^{(1)} x_2^2 \qquad (5.3)$$

The coefficients of the form $a_{ij}^{(1)}$ are chosen in such a fashion that the segments BB and C_1C_1 (direction of the vector c_1) are conjugate diameters of an ellipse $w_1 = N_1$. Now we pass through the ends of the segment BB an infinite number of ellipses $w_m = N_1$ (corresponding to various values of η), each of which has as conjugate diameters the segments BB and C_mC_m $(\eta_1 \leqslant \eta_m \leqslant \eta_2)$.

By changing the constant N_1 we obtain a new system of level lines $w(x, \eta_m) = \text{const}$, where one can verify that also for the new ellipses the general intersection points remain on the line $\beta_1 x_1 + \beta_2 x_2 = 0$.

We have for a fixed $\eta = \eta_1$

$$\frac{dw_1}{dt} = \frac{\partial w_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w_1}{\partial x_2}\frac{dx_2}{dt}$$

or because of (5.1)

$$\frac{dw_1}{dt} = (k + c\xi, \operatorname{grad} w_1) \qquad (5.4)$$

The vector grad w_1 has the direction of the exterior normal. In the neighborhood of the points B we obtain

$$\frac{dw_1}{dt} = (k, \operatorname{grad} w_1) = n(k, l_1) < 0 \qquad (n < 0) \qquad (5.5)$$

By means of a choice of the factor m in (5.2) with an increasing distance from the line $\phi(x_1, x_2) = 0$, one can reach a negative value of the derivative dw_1/dt, since the line $\phi(x_1, x_2) = 0$ separates the plane x_1x_2 into regions in which the scalar product $(c_1, \operatorname{grad} w_1)$ has opposite signs.

Let us construct a quadratic form $v(x, \eta)$, considering $\eta(t)$ to be a random quantity such that the coefficients of the form $v(x, \eta)$ for each value of $\eta = \eta_m$ are equal to the coefficients $a_{ij}^{(m)}$ in the expression for w_m of the form (5.3).

We compute the derivative of mathematical expectation $dM\{v\}/dt$ on the basis of Equation (5.1) (see Formula (2.3)):

$$\frac{dM\{v\}}{dt} = (k + c\,\xi, \mathrm{grad}_\eta\, v) + \int_{\eta_1}^{\eta_2} [v(x, \beta) - v(x, \eta)]\, d_\beta q(\eta, \beta) \qquad (5.6)$$

Here $\mathrm{grad}_\eta\, v$ is computed for a fixed η.

At the points of the line $\beta_1 x_1 + \beta_2 x_2 = 0$, which is the geometric locus of general intersection points of equal values of the form $v(x, \eta)$ for all $\eta_1 \leqslant \eta \leqslant \eta_2$, the integral on the right-hand side of (5.6) approaches zero.

At any other point of the plane $x_1 x_2$ which does not belong to the neighborhood of that line, a negative value of $dM\{v\}/dt$ can be assured by a choice of the multiplier m in (5.2). Thus, the sufficient condition for the ability to construct an allowable control in the case studied is the existence of a region in the phase plane $\{x_1, x_2\}$ (in the illustration this region is shaded) in which for $\eta_1 \leqslant \eta(t) \leqslant \eta_2$ the scalar product (k, l) is negative and the vector l is pointed in the direction of this region. These conditions will be, of course, sufficient for the system (5.1) and for the purpose of constructing the optimum control program.

We omit the discussion on the possibility of constructing the control program for the case of the action of the disturbance $\gamma(x)$, since this repeats completely the presentation of Section 2, subsection 2.

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